Representing totally disconnected, locally compact groups by countable structures

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Totally Disconnected Locally Compact Groups: Local to Global

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Polish groups, non-Archimedean groups

Begin with background. All topological groups G in this talk will be Polish. Note that each open subgroup is closed, and has countable index in G.

G is non-Archimedean if it has a basis of neighbourhoods of the identity consisting of open subgroups. Such groups have a basis of clopen sets.

They are, up to homeomorphism, the closed subgroups of the topological group $\mathrm{Sym}(\mathbb{N})$ of permutations of $\mathbb{N}$ with the usual topology of pointwise convergence.

(Locally) Roelcke precompact groups

Let G be a closed subgroup of $\mathrm{Sym}(\mathbb{N})$. Note G is compact iff each open subgroup has only finitely many (left) cosets.

Definition (for such G)

- G is Roelcke precompact (R.p.) if each open subgroup U has only finitely many double cosets. That is, there is finite $\alpha \subseteq G$ such that $G = U\alpha U$.
- G is locally Roelcke precompact if G has a Roelcke precompact open subgroup.

Borel classes of closed subgroups of $\mathrm{Sym}(\mathbb{N})$

The closed subgroups G of $\mathrm{Sym}(\mathbb{N})$ form a "standard Borel space":

- If σ is a string let $[\sigma] = \{\pi \in \mathrm{Sym}(\mathbb{N}) \colon \sigma \prec \pi\}$.
- The σ-algebra of Borel sets is generated by the sets

$$\{G \colon G \cap [\sigma] \neq \emptyset\}.$$

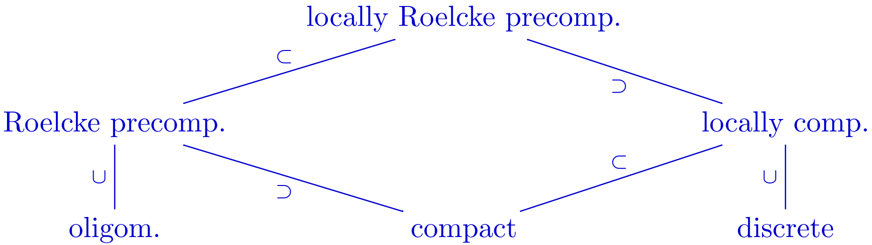
Kechris, N. and Tent, 2018; Logic Blog 2020:

Programme

(a) Determine whether classes $\mathcal{C}$ of closed subgroups of S_∞ are Borel.

(b) If $\mathcal{C}$ is Borel, study the relative complexity of the topological isomorphism relation using Borel reducibility $\leq_B$.

Some classes, and inclusion relations



- Isomorphism relation on each class in the diagram is $\leq_B$ graph isomorphism (Kechris, N. and Tent, 2018).
- $\cong$ on the profinite groups is $\geq_B$ graph isomorphism (Kechris, N. and Tent, 2018).
- $\cong$ on the class of oligomorphic groups is $\leq_B$ a countable Borel equivalence relation (N., Schlicht and Tent, 2021).

Two goals in this talk

(1) Show how to represent locally Roelcke precompact groups by certain countable structures called "coarse groups". Establish Borel duality.

(2) For tdlc groups, introduce a variant of the coarse groups called "meet groupoids". They are algebraically more concise, and hence can be used for an algorithmic theory of such groups.

I. Borel duality between locally Roelcke precompact groups and their coarse groups

Roelcke precompactness

In the first part of the talk, by G we always denote a closed subgroup of $\mathrm{Sym}(\mathbb{N})$. For such G general definitions of Roelcke, 1988; Rosendal, Zielinski, 2020 amount to this:

Definition

G is Roelcke precompact (R.p.) if each open subgroup U has only finitely many double cosets.
G is locally Roelcke precompact if it has a R.p. open subgroup.

Let T_∞ be the undirected tree with each vertex of infinite degree.

- $\mathrm{Aut}(T_\infty)$ is locally R.p. (Zielinski), and not locally compact.
- The stabiliser of a vertex is Roelcke precompact.

Some coarse language

The following was introduced for R.p. in Kechris, N., Tent 2018.

Given a locally R.p. G, let $\mathcal{M}(G)$ be its coarse group:

- The domain consists of (numbers encoding) the R.p. open cosets in G.
- Ternary relation "$AB \subseteq C$" on the domain.

- R.p. open cosets approximate elements of G, so this ternary relation approximates the binary group operation.
- Each R.p. open subgroup of G is a finite union of double cosets of a basic open subgroup. So $\exists$ only countably many such subgroups.
- Using descriptive set theory, we can view the operator $\mathcal{M}$ as a Borel function from locally R.p. groups to structures with domain $\mathbb{N}$.

Borel duality theorem

- An abstract coarse group is a structure on $\mathbb{N}$ with a ternary relation satisfying certain axioms[1]. Denote elements by A, B, C, and write the ternary relation suggestively as $AB \sqsubseteq C$.

- Let **CG** be the closure under isomorphism of the range of $\mathcal{M}$, among structures on $\mathbb{N}$ with one ternary relation.

- We aim at a duality

$$\text{locally R.p. groups } \mathbf{LRP} \underset{\mathcal{G}}{\overset{\mathcal{M}}{\rightleftarrows}} \mathbf{CG} \text{ class of coarse groups.}$$

[1] see N., Schlicht, Tent JML 2021, Coarse groups, and the isomorphism problem for oligomorphic groups, Def 2.1 onwards

Defining the reverse operation $\mathcal{G}$: from coarse groups to locally R.p. groups

Recall

- $\mathcal{M}(G)$ is the coarse group of a locally R.p. G,
- **CG** is the closure under isomorphism of the range of $\mathcal{M}$.

Definition

Given a structure $M \in \mathbf{CG}$, let $\mathcal{G}(M)$ be the closed subgroup of $\mathrm{Sym}(\mathbb{N})$ consisting of the permutations p such that

$$AB \sqsubseteq C \Longleftrightarrow p(A)B \sqsubseteq p(C) \text{ for each } A, B, C \in M.$$

Borel duality theorem

We have defined maps $\mathbf{LRP} \underset{\mathcal{G}}{\overset{\mathcal{M}}{\rightleftarrows}} \mathbf{CG}$.

Theorem

- $\mathbf{CG}$ is a Borel class. $\mathcal{M}$ and $\mathcal{G}$ are Borel maps.
- $\mathcal{M}$ and $\mathcal{G}$ are inverses up to isomorphism:
  For each $G \in \mathbf{LRP}$ and each $M \in \mathbf{CG}$,
  $$\mathcal{G}(\mathcal{M}(G)) \cong_{top} G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M.$$

The pair of functors $\mathcal{M}, \mathcal{G}$ yields an equivalence of the categories:

- **LRP** locally R.p. groups with topological isomorphism
- **CG** corresponding coarse groups with isomorphism of structures.

II. Computable duality for

totally disconnected,

locally compact (tdlc) groups

Basic fact on tdlc groups

Van Dantzig's theorem (1936): Each tdlc group G has a basis of neighbourhoods of 1 consisting of compact open subgroups. In particular, if G is countably based it is non-Archimedean.

So tdlc groups form a proper subclass of the non-Archimedean locally R.p. groups.

van Dantzig follows from these two facts:

- For each totally disconnected, locally compact space, the clopen sets form a basis.
- For each Hausdorff group, each compact open neighbourhood of 1 contains a compact open subgroup.

Some examples of tdlc groups G

- All profinite groups and all discrete groups.
- $(\mathbb{Q}_p, +)$, the additive group of p-adic numbers for a prime p.
- The semidirect product $\mathbb{Z} \ltimes \mathbb{Q}_p$ where $g \in \mathbb{Z}$ acts as $x \mapsto xp$ on $\mathbb{Q}_p$.
- The groups $\mathrm{SL}_n(\mathbb{Q}_p)$ for $n \geq 2$.
- $\mathrm{Aut}(T_d)$, the automorphisms of an undirected tree with each vertex of degree d. Stabilizer of a vertex is a compact open subgroup.

Motivating questions

(A) How can one define a computable presentation of a tdlc group? Which tdlc groups have such a presentation?

(B) Given a computable presentation of a tdlc group, are objects such as the (rational-valued) Haar measures, the modular function, or the scale function computable?

Computable duality

We will provide two kinds of computable presentation: one based on paths on trees, the other on a variant of coarse groups.

They turn out to be equivalent, in the sense that from a presentation of one type one can construct a presentation of the other type.

Defining computably tdlc groups

via Baire presentation

Trees, and computable function on their paths

$\mathbb{N}^*$ denotes tuples of natural numbers, pictured as a directed tree. For a subtree T of $\mathbb{N}^*$ without leaves, by $[T]$ one denotes its set of paths.

$F\colon [T] \to [T]$ is computable if there is a Turing machine L as follows. If α is on the read-only input tape, it puts $F(\alpha)$ on the write-only output tape.

For example, let $F(\alpha)(n) = \sum_{i \leq n} \alpha(n)$. This F is computable.

$$\text{computable} \Rightarrow \text{continuous}$$

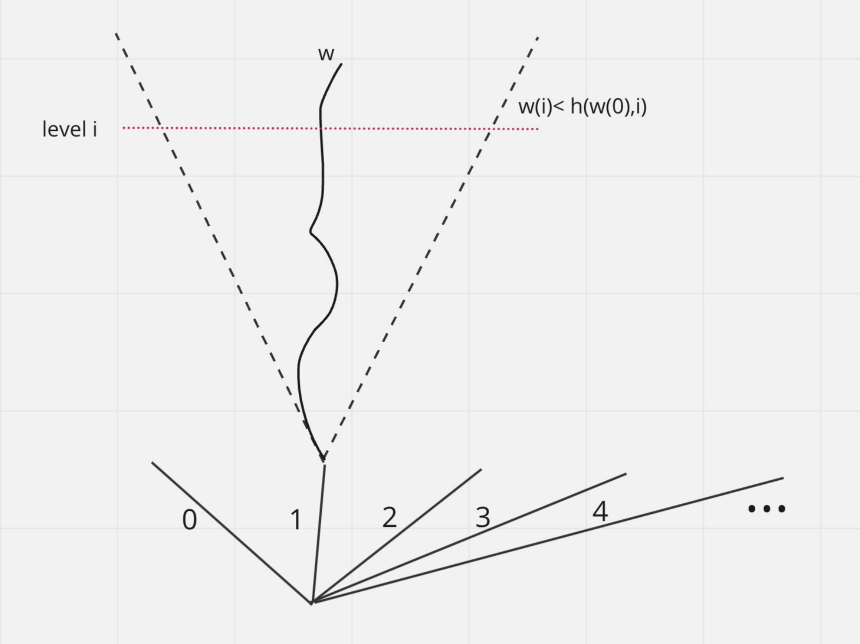
because such a TM L determines any entry on the output tape from finitely many queries to the input tape. E.g., a function M with $M(\alpha)(0) = \min\{\alpha(i) \colon i \in \mathbb{N}\}$ is not computable.

Computable Baire presentation of tdlc G

We use that each 0-dimensional Polish space is homeomorphic to some path set $[T]$.

- The domain of this presentation of G equals $[T]$ for a computable subtree of $\mathbb{N}^*$ such that
  - the only possible infinite branching is at the root
  - there is a computable bound $h\colon \mathbb{N} \to \mathbb{N}$ such that $w(i) \leq h(i, w(0))$ for each $w \in T$ and $i > 0$.
    (The tree above n is finitely branching, effectively in n.)

- The operations of G are computable.


w
w(i)< h(w(0),i)
level i
0
1
2
3
4
...

$\mathbb{F}_p((t))$ and $\text{Aut}(T_d)$ have a comp. Baire presentation

Let Q be the tree of strings $\tau \in \mathbb{N}^*$ such that

- all entries, except possibly the first, are among $\{0, \ldots, p-1\}$,
- $r0 \not\preceq \tau$ for each $r > 0$.

String $r\sigma \in Q$ denotes the Laurent polynomial

$$x^{-r} \sum_{0 \leq k < |\sigma|} \sigma(k) x^k.$$

One checks that addition (and also multiplication) on $\mathbb{F}_p((t))$ are computable. $\mathbb{Q}_p$ works in a similar way. Then we can show that $GL_n(\mathbb{F}_p((t)))$ and $GL_n(\mathbb{Q}_p)$ are computably tdlc.

For $\text{Aut}(T_d)$, the bottom level of T tells where a fixed vertex v of T_d goes. The k-th level says where vertices at distance k from v go.

Computability in the abelian case

Theorem (Lupini, Melnikov and N., J Algebra, 2022)

Let G be an abelian tdlc group. The following are equivalent.

(1) G is computably tdlc.

(2) There exist:

(i) a computable profinite group K

(ii) a computable discrete group L

such that G is a topological extension of L by K via a computable co-cycle $c\colon L \times L \to K$.

(3) A characterisation as a certain computable inverse limit of countable groups.

Defining computably tdlc groups via meet groupoids

What's a groupoid? (Old notion)

Intuitively, the notion of a groupoid generalizes the notion of a group by allowing that the binary operation is partial.

- A groupoid is given by a domain $\mathcal{W}$ on which a unary operation $(.)^{-1}$ and a partial binary operation, denoted by "$\cdot$", are defined.
- Category view: a groupoid is a small category in which each morphism has an inverse.
- $A\colon U \to V$ means that U, V are idempotent ($U \cdot U = U$), and $A = UA = AV$.

The meet groupoid of a tdlc group G

- $\mathcal{W}(G)$ is an algebraic structure on the countable set of compact open cosets in G, together with $\emptyset$.
- This structure is a partially ordered groupoid. The partial order is set inclusion. We can multiply a left coset A of some subgroup U with a right coset B of the same U. This is a coset because, if $A = aU$ and $B = Ub$ some $a, b \in G$, then
$$AB = aUb = U^{a^{-1}}ab = abU^b.$$
- The intersection of two compact open cosets is either empty or is such a coset itself. So $\mathcal{W}(G)$ is a meet semilattice.

One can define in a first-order way the coarse group from the meet groupoid, and conversely. However, they are not computationally equivalent because the definitions need a lot of quantifiers (which amount to unbounded search over the structure).

Abstract definition of a groupoid

Definition

A meet groupoid is a groupoid $(\mathcal{W}, \cdot, (.)^{-1})$ that is also a meet semilattice $(\mathcal{W}, \cap, \emptyset)$ of which $\emptyset$ is the least element.
Writing $A \subseteq B \iff A \cap B = A$, it satisfies the conditions

- $\emptyset^{-1} = \emptyset = \emptyset \cdot \emptyset$, and $\emptyset \cdot A$ and $A \cdot \emptyset$ are undefined for each $A \neq \emptyset$,
- if U, V are idempotents such that $U, V \neq \emptyset$, then $U \cap V \neq \emptyset$,
- $A \subseteq B \iff A^{-1} \subseteq B^{-1}$, and
- if $A_i \cdot B_i$ are defined $(i = 0, 1)$ and $A_0 \cap A_1 \neq \emptyset \neq B_0 \cap B_1$, then

$$(A_0 \cap A_1) \cdot (B_0 \cap B_1) = A_0 \cdot B_0 \cap A_1 \cdot B_1.$$

Automorphism group of G and Chabauty space

Let $\mathcal{W} = \mathcal{W}(G)$ be the meet groupoid of a tdlc group G.

Proposition (Melnikov and N. '22; Logic Blog '23)

- $\mathrm{Aut}(G)$ with the usual Braconnier topology is canonically homeomorphic to $\mathrm{Aut}(\mathcal{W})$:
  Send $\phi \in \mathrm{Aut}(G)$ to its action on $\mathcal{W}$.

- The Chabauty space $\mathcal{S}(G)$ of closed subgroups of G can be canonically represented by a closed subset of $2^{\mathcal{W}}$, consisting of certain ideals of $\mathcal{W}$.

Computably tdlc groups via meet groupoids

A meet groupoid $\mathcal{W}$ is called Haar computable if

(a) its domain is a computable subset D of $\mathbb{N}$;

(b) the groupoid and meet operations are computable; in particular, the relation $\{\langle x, y\rangle \colon x, y \in D \wedge x \cdot y \text{ is defined}\}$ is computable;

(c) the function sending a pair of idempotents $U, V \in \mathcal{W}$ to the number of (left, say) cosets of $U \cap V$ in U is computable.

Definition (Computably tdlc groups via meet groupoids)

Let G be a tdlc group. We say that G is computably tdlc via a meet groupoid if $\mathcal{W}(G)$ has a Haar computable copy $\mathcal{W}$.

Computable duality

Theorem

A group G is computably tdlc via a Baire presentation $\iff$ G is computably tdlc via a meet groupoid.
From a presentation of G of one type, one can uniformly obtain a presentation of G of the other type.

As a corollary to the proof, we have:

Corollary

- Let $\mathcal{W}$ be a Haar computable copy of $\mathcal{W}(G)$ (with domain $\mathbb{N}$).
- The left and right actions $[T] \times \mathbb{N} \to \mathbb{N}$, given by

$$(g, A) \mapsto gA \text{ and } (g, A) \mapsto Ag,$$

  are computable.

Algorithmic properties of objects associated with a tdlc group

The modular function is computable

Throughout, let G be computably tdlc via a Baire presentation based on $[T]$, and let $\mathcal{W}$ be the Haar computable copy of $\mathcal{W}(G)$

- by definition, the modular function is $\Delta(g) = \mu(Ug)/\mu(U)$, where U is any compact open subgroup, μ a left Haar measure;
- we may assume μ is rational valued, and hence that μ is computable.

Since the right action of G on $\mathcal{W}$ is computable, we have:

Proposition

The modular function $\Delta \colon [T] \to \mathbb{Q}^+$ is computable.

The Cayley-Abels graphs are computable

If G is compactly generated, there is a compact open subgroup U, and a set $S = \{s_1, \ldots, s_k\} \subseteq G$ such that $S = S^{-1}$ and $U \cup S$ algebraically generates G. The Cayley-Abels graph

$$\Gamma_{S,U} = (V_{S,U}, E_{S,U})$$

of G is given as follows. The vertex set $V_{S,U}$ is the set $G \backslash U$ of left cosets of U, and the edge relation is

$$E_{S,U} = \{\langle gU, gsU \rangle \colon \ g \in G, s \in S\}.$$

Theorem

Suppose that G is computably tdlc and compactly generated. Each Cayley-Abels graph $\Gamma_{S,U}$ of G has a computable copy $\mathcal{L}$.

Algorithmic properties of the scale function

For a compact open subgroup V of G and an element $g \in G$ let

$$m(g, V) = |V^g : V \cap V^g|.$$

Recall the scale function $[T] \to \mathbb{N}$ is

$$s(g) = \min\{m(g, V) \colon V \text{ is a compact open subgroup}\}.$$

E.g., in $\mathbb{Z} \ltimes \mathbb{Q}_p$, where generator $g \in \mathbb{Z}$ acts as $x \mapsto xp$, we have $s(g) = 1$, $s(g^{-1}) = p$.

Fact

The scale function is computably approximable from above.

Example

For $d \geq 3$, the scale function on $\mathrm{Aut}(T_d)$ in the canonical computable presentation is computable.

A noncomputable scale

For the given examples the scale is computable. However:

Theorem (Melnikov, N., Willis, 2022)

There is a computable presentation of a tdlc group G based on a tree T such that the scale function $s\colon [T] \to \mathbb{N}$ is not computable.

In fact, there is a uniformly computable sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $s(g_n) = 2$ if $n \notin \mathcal{K}$ (the halting problem), and 1 otherwise.

Proof. We write H for the additive group of $\mathbb{F}_2((t))$.
We use the canonical computable Baire presentation $(Q, \mathrm{Mult}, \mathrm{Inv})$ of H: recall that string $r\sigma$ denotes the Laurent polynomial $x^{-r}\sum_{0\le k<|\sigma|}\sigma(k)x^k$.

Proof: c-bounded permutations

For $c \in \mathbb{N}$, we say that a permutation α of $\mathbb{Z}$ is c-bounded if $|\alpha(z) - z| \leq c$ for each $z \in \mathbb{Z}$.
Then the function $\widehat{\alpha}$ defined on H by

$$\widehat{\alpha}(\sum_{k \in \mathbb{Z}} r_k x^k) = \sum_{k \in \mathbb{Z}} r_{\alpha(k)} x^{\alpha(k)})$$

is a continuous automorphism of H.

Claim (easy)

Let α be a computable c-bounded permutation of $\mathbb{Z}$.
Then $\widehat{\alpha}\colon [Q] \to [Q]$ is computable, uniformly in c and the program for a Turing machine computing α.

Proof: encode $\mathcal{K}$ into the scale

Since the halting problem $\mathcal{K}$ is recursively enumerable, $\mathcal{K} = \bigcup_t \mathcal{K}_t$ for a suitable computable sequence of strong indices for finite subsets of $\mathbb{N}$. May assume $\mathcal{K}_t = \mathcal{K}_{t-1}$ for t odd. The following defines a computable function $\mathbb{N}^+ \times \mathbb{Z} \to \mathbb{Z}$ via $\langle i, t \rangle \mapsto \beta_i(t)$:

$$\beta_i(t) = \begin{cases} t+2 & \text{if } t \text{ is even and } i \notin \mathcal{K}_t \\ t-2 & \text{if } t \text{ is odd and } i \notin \mathcal{K}_t \\ t+1 & \text{if } t \text{ is even and } i \in \mathcal{K}_t - \mathcal{K}_{t-1} \\ t & \text{if } i \in \mathcal{K}_{t-1} \end{cases}$$

If $i \notin \mathcal{K}$ then β_i is the permutation of $\mathbb{Z}$ that adds 2 to even numbers, and subtracts 2 from odd numbers. So $s(\widehat{\beta_i}) = 2$.
If $i \in \mathcal{K}$, let t be least such that $i \in \mathcal{K}_t$. Then $s(\widehat{\beta_i}) = 1$ because the nontrivial cycle of β_i "turns around" at position t.

Proof: putting it together

Now let $G_i = \mathbb{Z} \ltimes_{\gamma_i} H$, where γ_i is the action $\mathbb{Z} \times H \to H$ given by $\gamma_i(z, h) = \overline{\beta}_i^z(h)$. This is computable uniformly in i.

Let g_i be the generator of $\mathbb{Z}$ in G_i whose conjugation action on H induces β_i. Then $s_{G_i}(g_i) = s_H(\widehat{\beta}_i)$ because G_i and H have the same compact open subgroups.

Let $G = (T, \mathrm{Mult}, \mathrm{Inv})$ be a computable Baire presentation of the local direct product $G = \bigoplus_{i \in \mathbb{N}^+}(G_i, U)$. It is clear that g_i viewed as an element of G is computable uniformly in i, and $s_G(\overline{g}_i) = s_{G_i}(g_i)$ for each i.

Thus $s_G(\overline{g}_i) = 1$ iff $i \in \mathcal{K}$, as required.

Quotients by computable closed normal subgroups

Theorem (Thm. 11.11 in '22 preprint with Melnikov)

Let N be a closed normal subgroup of G such that $Tree(N)$ is a computable subtree of $Tree(G)$. Then G/N is computably tdlc.

We prove this by building a Haar computable copy of the meet groupoid of G/N.
Along the way we have to show that "$\mathcal{K} \subseteq N\mathcal{L}$" is decidable, where $\mathcal{K}, \mathcal{L}$ are compact open sets.

Application: $PGL_n(\mathbb{Q}_p)$ is computably tdlc.

Outlook

- When is the scale computable?
  Find conditions on classes of tdlc groups implying this.
  For instance, we know that computable p-adic Lie groups have computable scale.
- Study "double coset scale" of $\pi \in \mathrm{Aut}(G)$, which is defined via counting double cosets. Perhaps this makes sense in the wider context of locally R.p. groups.
- In $\mathrm{Aut}(T_d)$, $3 \leq d \leq \infty$, we have $\mathrm{ds}(\pi) = 2$ where π is translation by one vertex along an axis (observed by Willis).
  In contrast, $s(\pi) = d - 1$ when $d < \infty$.

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